

MATHEMATICAL MODELING OF SCOTS PINE (*PINUS SILVESTRIS*) GROWTH PROCESS

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Abstract. There were formulated two general approaches to the problem under consideration indicating the experimental data sources. Mathematical symbols were introduced to denote each plant part and describe its function. There was constructed a discrete two – dimensional dynamical system that depicted Scots pine growth process followed by making suitable assumptions on its components. It has been found that this system is an iteration sequence of certain area transformation which served as the basis for the dynamical system development. Consequently, several important results like theorems, comments and charts were obtained. The obtained research results are related to variations of total dry weight M_n that grows exponentially, while linearly for n large enough. It was also demonstrated that the proportion of assimilatory to non assimilatory parts $\bar{\lambda}$ tends to the calculated stationary point λ_1 .

Key words: discrete dynamical system, stationary point, differential equation

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STUDY OBJECTIVE

The research objective is the modeling of growth process of Scots pine seedlings, whose effectiveness will be measured by changes in dry matter quantity, dependent on light intensity.

INTRODUCTION

Recently, quantitative studies of plant growth have become a matter of high interest. However, application of some popular growth functions used for both, plant and animal is not sufficient to perform its reliable analysis. While animals increase their body

weight by consumption of feed in a form of organic matter, plants facilitate their growth through a process occurring during photosynthesis. So, although plant growth relies on its all growing parts, a role of assimilatory organs is crucial due to significance of photosynthesis products partitioning into assimilatory and non-assimilatory parts.

Defining the problem in this way provides a reasonable simplicity to obtain effective analysis of this matter.

There are two basic research approaches to this problem, the first one subject to such factors like water, carbon dioxide, etc., and the other focused on productivity of assimilatory organs and partitioning of photosynthesis products into assimilatory and non-assimilatory parts. Importantly, the first approach is not satisfying in the efficiency because of the weather conditions that can change promptly so as a result, the second one has been taken into account.

The modelling of Scots pine growth process was based on the experimental data of accumulated mass and its distribution in small seedlings of this plant.

MATERIAL AND METHODS

The major mathematical tool employed for modelling the plant growth is the discrete two dimensional dynamical system. To match the model with the experimental data, the least squares method was used. The data were obtained upon the observation of Scots pine seedlings in the Laboratory of Physiomics and Crop Design, the Warsaw University of Life Sciences.

Let W_n and V_n denote dry matter of assimilatory and non-assimilatory parts, respectively.

It was found that the whole dry matter of plant was $M_n = W_n + V_n$ and proportion of both parts $\lambda_n = \frac{W_n}{V_n}$ and that $M_n, \lambda_n > 0$.

It was assumed [Szlenk and Żelawski 1985] that dynamics of plant growth process in time is depicted by the following differential equations system:

$$\begin{cases} W_{n+1} = W_n + \beta(\lambda_n, \delta) \alpha(W_n) W_n \\ V_{n+1} = V_n + (1 - \beta(\lambda_n, \delta)) \alpha(W_n) W_n \end{cases} \quad (1)$$

where: $\alpha(W_n)$ – a decreasing continuous function describing the pine needle length in n
– the period of time,

$\delta \in [0, 1]$ – describes influence of weather conditions on λ_n ($\delta = \text{const}$),

$\beta(\lambda_n, \delta)$ – a decreasing continuous function with relation to λ_n .

The above two functions show the following properties:

$$\lim_{W_n \rightarrow +\infty} \alpha(W_n) = 0, \quad \alpha(0) = \alpha_0 \quad (2)$$

$$\lim_{\lambda_n \rightarrow +\infty} \beta(\lambda_n, \delta) = 0, \quad \beta(\lambda_n, 0) = \alpha_0 \quad (3)$$

Consequently, $\alpha(W_n)W_n$ denotes the quantity of material produced by W_n in n – the period of time. $\beta(\lambda_n)$ indicates the amount of newly produced material used for assimilatory part enlargement. If $W_n \ll V_n$ then $\beta(\lambda_n) = 1$ which means that assimilatory part W_n absorbs nearly whole quantity of $\alpha(W_n)W_n$. If $W_n \gg V_n$ then $\beta(\lambda_n) \approx 0$ which means that non-assimilatory part V_n absorbs nearly whole quantity of $\alpha(W_n)W_n$.

Plant growth process described by the system (1) for $n = 0, 1, 2, \dots$ is the iteration sequence of the following surface transformation $(W, V) \rightarrow (\bar{W}, \bar{V})$:

$$\begin{cases} \bar{W} = W + \beta(\lambda)\alpha(W)W \\ \bar{V} = V + (1 - \beta(\lambda))\alpha(W)W \end{cases} \quad (4)$$

where: $\lambda = \frac{W}{V}$ and $M = W + V$ also $W, V > 0$. We intend to construct the discrete two dimensional dynamical system, hence $W = \lambda V$.

Its substitution into the formula on the whole plant dry matter: $M = \lambda V + V = (1 + \lambda)V$.

Facilitates V calculation to obtain $V = \frac{M}{1 + \lambda}$.

Substituting it into the formula on W , we have: $W = \frac{\lambda M}{1 + \lambda}$.

Taking into account the above two formulas and the system (4) we get:

$$\bar{M} = \bar{W} + \bar{V} = W + V + \alpha(W)W = M + \alpha\left(\frac{\lambda M}{1 + \lambda}\right)\left(\frac{\lambda M}{1 + \lambda}\right) := \bar{M}(M, \lambda) \quad (5)$$

$$\bar{\lambda} = \frac{\bar{W}}{\bar{V}} = \lambda \frac{1 + \beta(\lambda)\alpha\left(\frac{\lambda M}{1 + \lambda}\right)}{1 + (1 - \beta(\lambda))\alpha\left(\frac{\lambda M}{1 + \lambda}\right)} = \lambda f(M, \lambda) := \bar{\lambda}(M, \lambda) \quad (6)$$

The domain of the functions defined in (5) and (6) is area:

$D = \{(M, \lambda) : M > 0, \lambda > 0\}$ and they describe the transformation:

$\phi : D \rightarrow D; (M, \lambda) \rightarrow (\bar{M}, \bar{\lambda})$.

For $(M, \lambda) \in D$ is: $\phi(M, \lambda) = (\bar{M}(M, \lambda), \bar{\lambda}(M, \lambda)) \in D$.

Denoting: $\phi^n = \phi \circ \dots \circ \phi$ is:

$$(\bar{M}_n, \bar{\lambda}_n) = \phi(M_{n-1}, \lambda_{n-1}) = \phi^n(M, \lambda).$$

As a result, the dynamical system (D, ϕ) was obtained on the collection D which is a quarter of surface.

RESULTS AND DISCUSSION

The first obtained result is the fact that the proportion between the assimilatory parts and non-assimilatory ones $\bar{\lambda}$ in Scots pine tends to the preset value, which is equal to the stationary point λ_1 of the transformation ϕ . This result may be formulated as the theorem [Szlenk and Żelawski 1985] which is preceded by the following definition:

Definition 1: C^1 class is a collection of functions $f(x)$ which are continuous on the whole domain and have derivative $f'(x)$ in each point of the domain.

Theorem 2: It is assumed that functions $\alpha(W)$ and $\beta(\lambda)$ satisfy conditions (2) and (4.3) and belong to the collection of C^1 class. Then for all $M_n, \lambda_n > 0$:

$$\lim_{n \rightarrow +\infty} \bar{\lambda}(M_n, \lambda_n) = \lambda_1$$

where: λ_1 – stationary point.

An additional result obtained is the explicit formula for the stationary point λ_1 , depending only on environmental factors ratio δ [Szlenk and Żelawski 1985]. Namely, according to the stationary point definition, we take: $\bar{\lambda}(M, \lambda_1) = \lambda_1$. The solution of this equation is: $\lambda_1 = \frac{\beta(\lambda_1)}{1 - \beta(\lambda_1)}$. According to that, we assume such a simple shape of a function

$\beta(\lambda)$ which satisfies (3) and conditions from the Theorem 2: $\beta(\lambda) = \frac{\delta}{\delta + (1 - \delta)\lambda}$, where $0 < \delta < 1$. Besides, with such a definition of $\beta(\lambda)$ the experimental values of dry matter M_n are very close to the theoretical values [Szlenk and Żelawski 1985].

Taking into account that $\lambda_1 = \frac{W_1}{V_1}$, we obtain the following equations:

$$\beta(\lambda_1) = \frac{\delta V_1}{\delta V_1 + (1 - \delta)W_1}; \quad 1 - \beta(\lambda_1) = \frac{(1 - \delta)W_1}{\delta V_1 + (1 - \delta)W_1}$$

Substituting the above respective coefficients into the formula for λ_1 , we have:

$$\lambda_1 = \frac{\beta(\lambda_1)}{1 - \beta(\lambda_1)} = \frac{\delta V_1}{(1 - \delta)W_1} = \frac{\delta}{(1 - \delta)\lambda_1}$$

Calculating from the above λ_1 , we get:

$$\lambda_1 = \sqrt{\frac{\delta}{1 - \delta}}$$

The next result is formulated as a theorem [Szlenk and Żelawski 1985].

Theorem 3: If function describing the length of a leaf goes as follows:

$\alpha(W) = \frac{a_0}{a + W}$, where a_0, a – constants and if function $\beta(\lambda)$ belongs to the collection of a C^1 class then for sufficiently large n the whole dry matter M_n of Scots pine increases linearly.

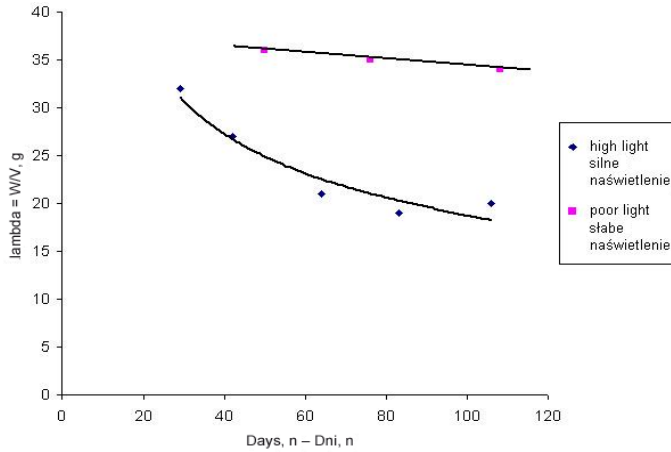


Fig. 1. Comparison of experimental trend and theoretical values for coefficient $\lambda = \frac{W}{V}$ in Scots pine seedlings grown under laboratory conditions with water under high and poor light intensity [Szlenk and Żelawski 1985]

Rys. 1. Porównanie trendu wartości eksperymentalnych i wartości teoretycznych dla współczynnika $\lambda = \frac{W}{V}$ w sadzonkach sosny zwyczajnej wznrastających w warunkach laboratoryjnych z udziałem wody przy silnym i słabym natężeniu światła [Szlenk i Żelawski 1985]

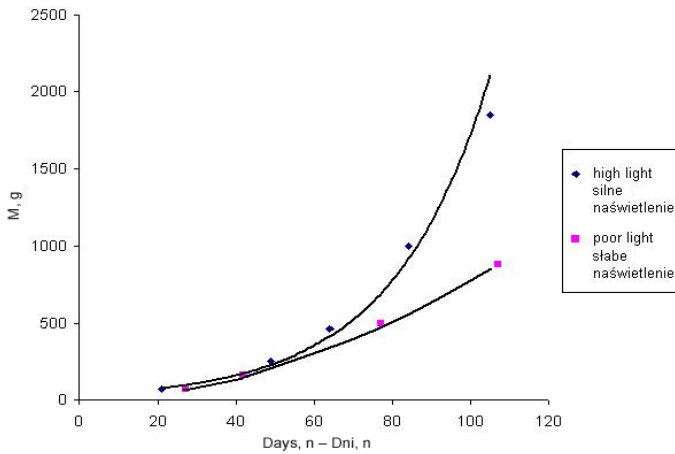


Fig. 2. Comparison of whole dry matter M growth of Scots pine in relation to time for theoretical and experimental values under high and poor light intensity [Szlenk and Żelawski 1985]

Rys. 2. Porównanie wzrostu całkowitej biomasy M sosny zwyczajnej dla wartości teoretycznych i eksperymentalnych przy silnym i słabym natężeniu światła [Szlenk i Żelawski 1985]

The below graphs obtained by the least squares method present the simulations of Scots pine growth dependent on changing light intensity.

One should mention about another popular growth models and their properties. Firstly is presented exponential model which has the following formula:

$$f(x) = \alpha + (\beta - \alpha) \exp\left(-\frac{x}{\gamma}\right)$$

where: α, β, γ – parameters.

Although this is not good for application, because it does not occur in real world.

Next model is the Gompertz model which has the following formula:

$$f(x) = \alpha \exp\left(-e^{-\kappa(x-\gamma)}\right)$$

where: α, γ, κ – parameters.

This equation could be applied for every single curve which describes a growth of a plant. Particularly could be applied allometric models which have a logarithmic formula as follows:

$$\ln M = c + \alpha \ln D$$

where:

- M – total biomass of a plant,
- D – stem diameter at the breast height of a Scots pine,
- c, α – parameters.

In this model correlation between M and D is high. Hence amount of biomass determines the diameter of a tree. There are some kinds of modifications of this model, for example:

$$\ln M = c + \alpha \ln D + \beta \ln H$$

where: H – tree height.

Since correlation between D and M has been high, the addition of H in this model only increased the correlation slightly and also reduced slightly the average deviation.

One could obtain another modification by changing variable H to another one:

$$\ln M = c + \alpha \ln D + \beta \ln W_d$$

where: W_d – wood density.

Adding wood density in the model is important in order to estimate the biomass for mixed species and big trees, since biomass estimates for larger D trees are more variable and have a disproportionately large contribution to forest biomass.

CONCLUSIONS

The obtained results of the coefficient λ allow to differentiate two extreme cases that have biological justification, i.e. when $\lambda = 0$ plant is devoid of assimilatory organs and λ

$= +\infty$ can denote a rootless shoot (for example a leaf or stem before rooting). It means that Scots pine tends to escape from the state where either assimilatory part W_n or non-assimilatory V_n are very much out of balance.

From the Theorem 2, a conclusion follows that growth curve of a plant has a shape similar to the letter “S” because according to previous argumentation the stationary point λ_1 , to which the limit of proportion $\bar{\lambda}$ tends, depends on the environmental conditions ratio δ which additionally have great impact on changes in a plant growth process.

The Theorem 3, which can have only theoretical meaning, by assumption of infinite growth concludes that for sufficiently large n the whole dry matter M_n is asymptotically equal to the linear function like: $f(n) = Pn$, where P – constant.

Interpretation of the above Figures implies that at high intensity of light over the time, the proportion λ declines. Whereas, at poor intensity of light, no marked changes are observable (Fig. 1).

Analysing the behaviour of the whole dry matter M_n , the conclusion is that over the time it increases exponentially 2-fold faster at high intensity of light as compared to poor intensity of light (Fig. 2).

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MODELOWANIE MATEMATYCZNE

PROCESU WZROSTU SOSNY ZWYCZAJNEJ (*PINUS SILVESTRIS*)

Streszczenie. Sformułowano dwa ogólne podejścia do rozważanego problemu, sygnalizując źródła danych eksperymentalnych. Wprowadzono symbolikę matematyczną dla każdej części rośliny i podano, jakie pełni funkcje. Skonstruowano dwuwymiarowy dyskretny układ dynamiczny opisujący proces wzrostu sosny zwyczajnej i poczyniono odpowiednie założenia odnośnie jego składowych. Zauważono, że układ ten jest ciągiem iteracji pewnego przekształcenia płaszczyzny, na bazie którego został utworzony układ dynamiczny. Dzięki niemu uzyskano kilka istotnych wyników w postaci twierdzeń, uwag i wykresów. Uzyskane wnioski dotyczą zachowania się całkowitej biomasy M_n , która ro-

śnie wykładniczo, a dla dostatecznie dużych n rośnie liniowo. Wnioski opisują również stosunek biomas $\bar{\lambda}_2$, który dąży do obliczonego punktu stałego λ_1 .

Słowa kluczowe: dyskretny układ dynamiczny, punkt stały, równanie różnicowe

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